THE DIFFERENCE PROPERTY AND HIGHER-ORDER CONVEXITY PROPERTIES OF DIFFERENCE OPERATORS

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Dedicated to the memory of Professor Roman Ger

ABSTRACT. We prove that the class of delta-n-convex functions has the difference property, which is a generalization of Ger's result on the difference property of the class of delta-convex functions [7]. Using this generalization, we prove that if for every non-negative number a the difference operator $\Delta_a f$ is $n-$convex or $n-$concave, for some function f, then $\Delta_a f$ is $n-$convex for all a or n-concave for every a. A connection of our results with Wright convexity of higher orders is also presented.

1. INTRODUCTION

For a function $f : \mathbb{R} \to \mathbb{R}$ and a number $a \geq 0$ the difference operator Δ_a is given by

$$\Delta_a f(x) = f(x + a) - f(x).$$

The problem that inspired this paper was proposed by the second author as Problem 4 in [22] and Problem 5 in [23] as follows.

Problem. Assume that $\Delta_a f$ is either strictly increasing or strictly decreasing for every $a > 0$. Is it strictly increasing for every $a > 0$ or strictly decreasing for every $a > 0$?

In [19], it was shown that the answer is positive if f is a continuous function. Balcerowski [1] proved that this is true in general.

Motivated by Balcerowski's result, the first author introduced in the paper [18] the following sets:

$$A_f = \{a \geq 0 : \Delta_a f \text{ is non-decreasing }\}, \tag{1}$$

$$B_f = \{a \geq 0 : \Delta_a f \text{ is non-increasing }\}. \tag{2}$$

In [18], the properties of the sets A_f, B_f were studied; among others, it was shown that A_f, B_f are closed subsemigroups of $[0, \infty)$. In consequence, it was observed that if

$$A_f \cup B_f = [0, \infty)$$

and

$$A_f \cap (0,\infty) \neq \emptyset, B_f \cap (0,\infty) \neq \emptyset,$$

then

$$A_f = B_f = [0,\infty).$$

Thus, Balcerowski's result was viewed from a more general perspective.

Now, observe a simple but interesting connection between the properties of sets A_f, B_f and the notion of Wright-convexity. We say that the function f is Wright-convex (W-convex) if it satisfies the inequality

$$f(tx + (1-t)y) + f((1-t)x + ty) \leq f(x) + f(y) \tag{3}$$

for all $x, y \in \mathbb{R}$ and $t \in [0,1]$. Wright first studied this inequality in [21], and Ng subsequently proved a fundamental result on the representation of Wright-convex functions in [11]. Namely, it was shown that f satisfies (3) if and only if it is of the form

$$f = a + g,$$

where $a : \mathbb{R} \to \mathbb{R}$ is an additive function and $g : \mathbb{R} \to \mathbb{R}$ is convex. See also [12], [13] for some more recent results connected with the Wright-convexity and [14] for an elementary proof of Ng's result. In the following remark, we note the connection of the properties of sets A_f and B_f with this notion.

Remark 1. It is easy to see that f is W-convex if and only if $A_f = [0,\infty)$.

At the end of the paper, we will compare the notion that we consider with the higher-order version of Wright convexity.

We explained above only one connection of the properties of the difference operator with a classical notion. However, the difference operator is a basic concept widely used in many branches of mathematics (discrete mathematics, numerical analysis, functional equations, and others). Therefore, it is of interest to study its various properties. A natural question is the generalization of Balcerowski's result to higher-order convex functions. We will explain it in detail in the next section of the paper.

2. The generalization of the original problem

In the previous section, we briefly described the results for the case where the function $\Delta_a f$ was increasing (or decreasing). However, monotonicity falls under the broader notion of higher-order convexity. Indeed, let $I \subset \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ be a function, and let the higher-order divided difference be defined by

$$f[x_0,\ldots,x_n] := \sum_{j=0}^{n} \frac{f(x_j)}{\prod_{k=0,k\neq j}^{n}(x_j-x_k)}.$$

Then we say that f is $n-$convex if

$$f[x_0,\ldots,x_n,x_{n+1}] \geq 0, x_1,\ldots,x_{n+1} \in I.$$

Thus, 0-convexity means that

$$\frac{f(x_0)-f(x_1)}{x_0-x_1} \geq 0$$

i.e., the function f is 0-convex if and only if it is increasing. Similarly, it is easy to see that 1-convexity is the standard convexity, and for $n>1$ we get new conditions which are called higher-order convexity. Higher-order convex functions were first studied by Popoviciu, who obtained the following characterization.

Theorem 2 (T. Popoviciu [15])**.** *Let $n \in \mathbb{N} \setminus \{1\}$. Then a function $f: I \to \mathbb{R}$ is n-convex if and only if it is $(n-1)$-times continuously differentiable and $f^{(n-1)}$ is convex on I.*

From the above theorem, it immediately follows that a function of the class C^{n+1} is $n-$convex if and only if $f^{(n+1)} \geq 0$. This simple observation shows that, indeed, higher-order convex functions naturally extend the standard convexity notion.

Now, we may extend the problem described above as follows: assume that $\Delta_a f$ is, for all a, either $n-$convex or $n-$concave. Is it true that $\Delta_a f$ is $n-$convex for all a or $n-$concave for all a? In the paper, we answer this question in the affirmative.

To this end, we introduce the generalized versions of sets A_f, B_f from the previous section. That is, let

$$A(f,n) = \{a \geq 0 : \Delta_a f \text{ is } n\text{-convex }\}$$

and

$$B(f,n) = \{a \geq 0 : \Delta_a f \text{ is } n\text{-concave }\}.$$

We will study the properties of these sets and their relationships.

3. Preliminaries

Let $BV(\mathbb{R})$ be the class of all functions $f: \mathbb{R} \to \mathbb{R}$ having bounded variation on any finite interval $I \subset \mathbb{R}$.

Remark 3. Recall that it is well known that a function $f: \mathbb{R} \to \mathbb{R}$ belongs to the set $BV(\mathbb{R})$ if and only if $f = f_1 - f_2$, where the functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ are non-decreasing.

Let $\mathcal{M}(\mathbb{R})$ be the set of all signed measures on $\mathcal{B}(\mathbb{R})$ that are finite on compact sets. It is not difficult to show that functions from $BV(\mathbb{R})$ are the distribution functions of the signed measures from $\mathcal{M}(\mathbb{R})$ (see [18]).

Take a function f from $BV(\mathbb{R})$ and, for simplicity, assume that f is left continuous. Then $f = F_\mu$ for some $\mu \in \mathcal{M}(\mathbb{R})$ such that

$$\mu\big([a,b)\big) = f(b) - f(a), \quad a, b \in \mathbb{R}; a < b. \tag{4}$$

Moreover, the left continuous distribution function F_μ corresponding to μ is uniquely determined (up to a constant), see [18].

By Theorem 2 and using classical facts concerning convex functions (see, for example, [9]), we can obtain the following characterization of n-convexity. Here, $f_-^{(n)}$ is the left-hand nth derivative of f.

Proposition 4. *Let $I \subset \mathbb{R}$ be an open interval, a function $f : I \to \mathbb{R}$ is $n-$convex ($n-$concave) if and only if $f_-^{(n)}$ exists and is non-decreasing (non-increasing) and is left continuous on I.*

We recall the notion of delta-n-convexity (see, for example, [17]). A function is said to be delta-n-convex if it may be written as a difference of two $n-$convex functions. In view of Remark 3, the functions of bounded variation may be called delta-0-convex. In addition, delta-1-convex functions have been extensively studied in [20].

Now, we recall the integral representation of n-convex functions, obtained in [16]. Note that here $u_+ = \max(u, 0)$.

Theorem 5. *Let c, d be such that $-\infty \le c < d \le \infty$ and let $\xi \in (c, d)$. A function $f : (c, d) \to \mathbb{R}$ is n-convex if and only if*

$$\begin{aligned} f(x) = &\int_{(c,\xi)} (-1)^{n+1} \frac{[-(x-u)]_+^n}{n!} d\mu(u) \\ &+ \int_{[\xi,d)} \frac{(x-u)_+^n}{n!} d\mu(u) + Q_{\xi,n}(x), \end{aligned} \tag{5}$$

where μ is a Borel non-negative measure on $\mathcal{B}((c,d))$ that is finite on compact sets and $Q_{\xi,n}$ is a polynomial of degree at most n.

Moreover, μ is the measure with the distribution function $F_\mu = f_-^{(n)}$ i.e. for all $a, b \in (c, d)$ with $a < b$, we have

$$\mu\big([a,b)\big) = f_-^{(n)}(b) - f_-^{(n)}(a). \tag{6}$$

Remark 6. It is clear that the function f is n-concave (delta-n-convex) if and only if its integral representation is given by (5), where μ is the non-positive Borel measure (the signed Borel measure) that is finite on compact sets such that (6) is satisfied.

In the following propositions, we give the following characterizations of delta-n-convexity.

Proposition 7. *Let $I \subset \mathbb{R}$ be an open interval, a function $f : I \to \mathbb{R}$ is delta-n-convex (delta-n-concave) if and only if $f_-^{(n)}$ exists and $f_-^{(n)} \in BV(I)$.*

Proof. Let f be a delta-n-convex function . Then by the definition of delta-n-convex functions, $f = f_1 - f_2$, where f_1, f_2 are n-convex functions. By Proposition 4, $(f_1)_-^{(n)}, (f_2)_-^{(n)}$ are non-decreasing functions, which implies that $f_-^{(n)} = (f_1)_-^{(n)} - (f_2)_-^{(n)} \in BV(I)$.

Assume now that f is a function such that $f_-^{(n)}$ exists and $f_-^{(n)} \in BV(I)$. Let g be the function given by (5) with the signed Borel measure μ such that $F_\mu = f_-^{(n)}$, $\xi \in I$ and $Q_{\xi,n} = 0$ Then, by Remark 6, the function g is delta-n-convex such that $F_\mu = g_-^{(n)} = f_-^{(n)}$. Since $g_-^{(n)} = f_-^{(n)}$, there exists a polynomial of degree at most n, say Q, such that $f = g + Q$. Taking into account that g is delta-n-convex, there exist n-convex functions g_1, g_2 such that $g = g_1 - g_2$. We put $f_1 = g_1 + Q$, $f_2 = g_2$. Thus, we obtain that $f = f_1 - f_2$, where f_1, f_2 are n-convex functions, which implies that f is delta-n-convex. The proof is complete. □

By Proposition 7, we can obtain the following characterizations of delta-n-convexity.

Proposition 8. *Let $I \subset \mathbb{R}$ be an open interval, a function $f : I \to \mathbb{R}$ is delta-n-convex (delta-n-concave) if and only if is $(n-1)$-times continuously differentiable and $f^{(n-1)}$ exists and $f^{(n-1)}$ is delta-convex on I.*

Furthermore, since the 0-convex (0-concave) functions are non-decreasing (non-increasing), we have

$$A(f, 0) = A_f$$

and

$$B(f, 0) = B_f,$$

where A_f and B_f are given by (1) and (2), respectively.

Let $\mathcal{S}$ be the set of all closed additive subsemigroups of $[0, \infty)$ containing 0. In [18] the following results were obtained for the sets A_φ, B_φ.

Theorem 9. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a function, if $\varphi \in BV(\mathbb{R})$ then $A_\varphi, B_\varphi \in \mathcal{S}$.*

Theorem 10. *Let $S \in \mathcal{S}$. Then, there exists $\varphi \in BV(\mathbb{R})$ such that $A_\varphi = S$ $(B_\varphi = S)$.*

Theorem 11. *Let $a \geq 0$ be a number, let $\mu \in \mathcal{M}(\mathbb{R})$ and $\varphi = F_\mu$. Then*

a) $a \in A_\varphi$ if and only if $\mu(B + a) \geq \mu(B)$ for all $B \in \mathcal{B}(\mathbb{R})$,
b) $a \in B_\varphi$ if and only if $\mu(B - a) \geq \mu(B)$ for all $B \in \mathcal{B}(\mathbb{R})$.

Theorem 12. *Let $\varphi \in BV(\mathbb{R})$. Then one of the following conditions is fulfilled:*

a) $A_\varphi = B_\varphi = \{0\}$,
b) $A_\varphi = \{0\}$, $B_\varphi \cap (0, \infty) \neq \emptyset$,
c) $A_\varphi \cap (0, \infty) \neq \emptyset, B_\varphi = \{0\}$,
d) there exists $a_0 > 0$ such that

$$A_\varphi = B_\varphi = \{ja_0 : j = 0, 1, 2, \dots\},$$

e) $A_\varphi = B_\varphi = [0, \infty)$.

In the next section, we will study the properties of the sets $A(f, n)$ and $B(f, n)$. The connections with delta-n-convex functions will be exhibited.

4. Results for delta-n-convex functions.

In the first result of this section, we show that the set $A(f, n)$ is equal to the set $A(f_-^{(n)}, 0)$. It is a result in the spirit of Theorem 2 and will allow us to use the known monotonicity properties to work with the higher-order case.

Theorem 13. *Let the function $f : \mathbb{R} \to \mathbb{R}$ be delta-n-convex. Then we have $f_-^{(n)} \in BV(\mathbb{R})$ and*

$$A(f, n) = A(f_-^{(n)}, 0), \tag{7}$$

$$B(f, n) = B(f_-^{(n)}, 0). \tag{8}$$

Proof. Let f be a delta-n-convex function. Using Proposition 7, we know that $f_-^{(n)}$ exists and belongs to $BV(\mathbb{R})$.

Let $a > 0$, by the definition of $A(f, n)$ we know that $a \in A(f, n)$ if and only if $\Delta_a f$ is n-convex. Further, by Proposition 4, $\Delta_a f$ is n-convex if $(\Delta_a f)_-^{(n)}$ is non-decreasing. We have

$$(\Delta_a f)_-^{(n)}(x) = \big(f(x+a) - f(x)\big)_-^{(n)} = f_-^{(n)}(x+a) - f_-^{(n)}(x) = \Delta_a f_-^{(n)}(x).$$

Thus, $\Delta_a f$ is n-convex if and only if $a \in A(f_-^{(n)}, 0)$ and (7) is proved, similarly we can prove (8). $\square$

Proposition 14. *Let $\varphi \in BV(\mathbb{R})$ and let n be a positive integer. Then there exists a delta-n-convex function $f : \mathbb{R} \to \mathbb{R}$ such that*

$$A(\varphi, 0) = A(f, n). \tag{9}$$

Proof. Let $\varphi \in BV(\mathbb{R}), \xi \in \mathbb{R}$ and let μ be the signed measure on $\mathcal{B}(\mathbb{R})$ such that $F_\mu = \varphi$. Let f be the delta-n-convex function given by (5) with $Q_{\xi,n} = 0$. By Theorem 5 and Remark 6

$$f_-^{(n)} = F_\mu = \varphi.$$

By Proposition 13,

$$A(f, n) = A(f_-^{(n)}, 0),$$

consequently, (9) is satisfied. □

Proposition 15. *Let n be a positive integer and f be a delta-n-convex function. Then $A(f, n), B(f, n) \in \mathcal{S}$.*

Proof. Let f be a delta-n-convex function. By Theorem 13, we know that $f_-^{(n)} \in BV(\mathbb{R})$ and

$$A(f, n) = A(f_-^{(n)}, 0),$$

while by Theorem 9, we have

$$A(f_-^{(n)}, 0) \in \mathcal{S},$$

which finishes the proof. The case of $B(f, n)$ is similar. □

Proposition 16. *Let $S \in \mathcal{S}$ and let n be a positive integer, then there exists a delta-n-convex function f such that $A(f, n) = S$.*

Proof. By Theorem 10, there exists $\varphi \in BV(\mathbb{R})$ such that

$$A_\varphi = A(\varphi, 0) = S.$$

On the other hand, by Proposition 14, there exists a delta-n-convex function f such that $A(f, n) = A(\varphi, 0)$, which ends the proof. □

Proposition 17. *Let n be a positive integer and let $a \in (0, \infty)$. If $f : \mathbb{R} \to \mathbb{R}$ is a delta-n-convex function and μ is the signed Borel measure satisfying $F_\mu = f_-^{(n)}$, then:*

a) $a \in A(f, n)$ if and only if $\mu(B + a) \geq \mu(B)$ for all $B \in \mathcal{B}(\mathbb{R})$,
b) $a \in B(f, n)$ if and only if $\mu(B - a) \geq \mu(B)$ for all $B \in \mathcal{B}(\mathbb{R})$.

Proof. Using Theorem 13, we have

$$A(f, n) = A(f_-^{(n)}, 0).$$

In view of Theorem 11, this means that the condition a) is satisfied. The proof of b) is analogous. □

Proposition 18. *Let n be a positive integer and let f be a delta-n-convex function. Then one of the following conditions is true:*

a) $A(f,n) = B(f,n) = \{0\}$,
b) $A(f,n) = \{0\}$, $B(f,n) \cap (0,\infty) \neq \emptyset$,
c) $A(f,n) \cap (0,\infty) \neq \emptyset, B(f,n) = \{0\}$,
d) there exists $a_0 > 0$ *such that*

$$A(f,n) = B(f,n) = \{ja_0 : j = 0, 1, 2, \dots\},$$

e) $A(f,n) = B(f,n) = [0,\infty)$.

Proof. Let f be a delta-n-convex function. By Theorem 13

$$A(f,n) = A(f_-^{(n)}, 0), B(f,n) = B(f_-^{(n)}, 0).$$

Now it is enough to use Theorem 12 to get that one of the conditions a)...e) is satisfied. □

Theorem 19. *Let n be a positive integer and f be a delta-n-convex function. If for every $a > 0$ $\Delta_a f$ is n-convex or $\Delta_a f$ is n-concave, then $\Delta_a f$ is n-convex for all $a > 0$ or $\Delta_a f$ is n-concave for all $a > 0$.*

Proof. If for every $a > 0$ $\Delta_a f$ is n-convex or n-concave, then we have

$$A(f,n) \cup B(f,n) = [0,\infty).$$

In view of Proposition 18, this means that one of the following conditions is satisfied:

(i) $A(f,n) = \{0\}, B(f,n) = [0,\infty)$,
(ii) $A(f,n) = [0,\infty), B(f,n) = \{0\}$,
(iii) $A(f,n) = B(f,n) = [0,\infty)$.

This implies that either $\Delta_a f$ is n-convex for all $a > 0$ or $\Delta_a f$ is n-concave for all $a > 0$. □

5. Generalization of Ger's result

In this section, we establish a result that allows us to drop the delta-n-convexity assumption from the previous section.

The starting point here is a result due to Ger, connected to the notion of the difference property. We say that the function class $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ has the *difference property* if the condition

$$\Delta_h f \in \mathcal{F}, h \in \mathbb{R}$$

implies that there exists an additive function A such that

$$f - A \in \mathcal{F}.$$

De Bruijn showed in [5] that the classes of continuous functions, functions that are k-times differentiable, C^k-functions, analytic functions,

functions which are absolutely continuous in any compact interval, and functions with bounded variation over any compact interval have this property. Then these results were generalized by several authors; see, for example, [3], [4], [6]. However, in our approach, we are interested in the following result of Ger.

Theorem 20 (R. Ger [7])**.** *Assume that for a function $f:\mathbb{R}\to\mathbb{R}$, its difference $\Delta_h f$ is delta-convex for each $h\in\mathbb{R}$. Then there exist a delta-convex function $g:\mathbb{R}\to\mathbb{R}$, and an additive function $A:\mathbb{R}\to\mathbb{R}$, such that*

$$f = g + A. \tag{10}$$

In the next result, we obtain the higher-order version of Theorem 20. Note that the proof is a modification of Ger's proof, and it is built upon the $\mathbb{Q}$-differentiability notion introduced in [2].

Theorem 21. *Let $n\in\mathbb{N}$. If $\Delta_h f$ is a delta-n-convex function for each h, then f may be represented in the form*

$$f = g + A, \tag{11}$$

where g is delta-n-convex and A is additive.

Proof. The proof is by induction. For $n=1$, the assertion is true because of Theorem 20.

Now, let $n\geq 2$ and assume that for each f such that $\Delta_h f$ is delta-$(n-1)$-convex we have (11). We will show that (11) holds for all functions such that $\Delta_h f$ is delta-n-convex.

Thus, let $\Delta_h f$ be a delta-n-convex function , i.e.

$$\Delta_h f(x) = \varphi_h(x) - \psi_h(x),$$

where functions φ_h, ψ_h are n-convex. Since $n\geq 2$, each $n-$convex function is of the class C^1, and C^1 functions have the difference property; therefore

$$f = b + A, \tag{12}$$

where b is in C^1 and A is additive. Now, let

$$d_{\mathbb{Q}} f(x,u) = \lim_{\mathbb{Q}^+\ni r\to 0} \frac{f(x+ru)-f(x)}{r}.$$

Note that the above limit exists because of (12). We have

$$\begin{aligned}
d_{\mathbb{Q}}&f(x+h,u) - d_{\mathbb{Q}}f(x,u)\\
&= \lim_{\mathbb{Q}^+\ni r\to 0} \frac{f(x+h+ru)-f(x+h)-f(x+ru)+f(x)}{r}\\
&= \lim_{\mathbb{Q}^+\ni r\to 0} \frac{f(x+h+ru)-f(x+ru)-\big(f(x+h)-f(x)\big)}{r}\\
&= \lim_{\mathbb{Q}^+\ni r\to 0} \frac{\Delta_h f(x+ru)-\Delta_h f(x)}{r}\\
&= \lim_{\mathbb{Q}^+\ni r\to 0} \frac{\varphi_h(x+ru)-\varphi_h(x)-\big(\psi_h(x+ru)-\psi(x)\big)}{r}\\
&= \varphi_h'(x)u - \psi_h'(x)u.
\end{aligned}$$

The functions $\varphi_h'(x), \psi_h'(x)$ are $(n-1)$-convex, thus for each u the function

$$x \mapsto \Delta_h d_{\mathbb{Q}} f(x,u)$$

is delta-$(n-1)$-convex. From the inductive hypothesis, we get

$$d_{\mathbb{Q}} f(x,u) = b_u(x) + A_u(x), \tag{13}$$

where b_u is a delta-$(n-1)$-convex function and A_u is additive. Define B_u by

$$B_u(x) = \int_0^x b_u(t)dt$$

and let

$$g_1(x) = f(x) - B_1(x) = b(x) + A(x) - B_1(x).$$

Using both of the above representations of g_1, and in view of (13), we get

$$d_{\mathbb{Q}} g_1(x,u) = b_u(x) + A_u(x) - B_1'(x)u$$

and

$$d_{\mathbb{Q}} g_1(x,u) = b'(x)u + A(u) - B_1'(x)u.$$

Taking $u = 1$, and equating the above forms of $d_{\mathbb{Q}} g_1(x,u)$ we get

$$A_1(x) = b'(x) + A(1) - b_1(x). \tag{14}$$

This means that $A_1(x) = \alpha x$ for some number α. Taking $\beta = -A(1)$, by (14) we get

$$b'(x) = b_1(x) + \alpha x + \beta. \tag{15}$$

Since b is of class C^1, we have

$$b(x) = b(0) + \int_0^x b'(t)dt \tag{16}$$

and, since b_1 is delta-$(n-1)$-convex we may write

$$b_1 = i_1 - i_2, \tag{17}$$

where i_1, i_2 are $(n-1)$-convex. Thus, from (15), (16) and (17), we have

$$b(x) = b(0) + \int_0^x i_1(t)dt - \int_0^x i_2(t)dt + \alpha x^2/2 + \beta x, \tag{18}$$

with some $\alpha, \beta \in \mathbb{R}$. Observe that the functions:

$$c_1(x) = \int_0^x i_1(t)dt + b(0) + \frac{\alpha x^2}{2} + \beta x$$

and

$$c_2(x) = \int_0^x i_2(t)dt$$

are n-convex. Moreover, using the representation (18) in (12), we obtain

$$f(x) = c_1(x) - c_2(x) + A(x)$$

and, since $c_1 - c_2$ is a delta-n-convex function, the proof is complete. □

6. A general result

Now we can prove the main result of the paper.

Theorem 22. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and let n be a positive integer. Then the following statements are equivalent:*

a) for every $a > 0$ the function $\Delta_a f$ is n-convex or n-concave,
b) for every $a > 0$ the function $\Delta_a f$ is n-convex or for every $a > 0$ the function $\Delta_a f$ is n-concave.

Proof. We only need to prove $a) \Rightarrow b)$. Thus, assume that *a)* is satisfied. Then obviously for each $a > 0$, the function $\Delta_a f$ is delta-n-convex. Now, from Theorem 21, we get

$$f = g + A \tag{19}$$

where g is delta-n-convex and A is additive. We have

$$\Delta_a f(x) = \Delta_a g(x) + A(a). \tag{20}$$

Thus, for every $a > 0$, the function $\Delta_a g$ is either n-convex or n-concave. In view of Theorem 19, this means that $\Delta_a g(x)$ is n-convex for every a or n-concave for every a. Taking into account (20), we obtain that the condition *b)* is satisfied. □

In the next corollary, we observe that we, in fact, know the exact form of the functions that we are dealing with.

Corollary 23. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and let n be a positive integer. Then the following statements are equivalent:*

a) for every $a > 0$ the function $\Delta_a f$ is n-convex or n-concave,
b) the function f is of the form

$$f = g + A,$$

where A is an additive function and g is an $(n+1)$-convex or $(n+1)$-concave function.

Proof. To prove the implication $a) \Rightarrow b)$, it is enough to check the proof of Theorem 22 carefully. Indeed, we get the condition $b)$ from (19) and from the fact that $\Delta_a g$ is either n-convex or n-concave. The other implication is obvious. $\square$

7. A connection with the higher-order Wright convexity

Now, we can get the aforementioned connection with higher-order Wright convexity. Let $I \subset \mathbb{R}$ be an interval, we say that a function f is Wright convex of order n if it satisfies the inequality

$$\Delta_{a_1,\dots,a_{n+1}} f(x) \geq 0,\ a_1, \dots, a_{n+1} > 0\ x \in I \cap \big(I - (a_1 + \cdots + a_{n+1})\big)$$

(see, for example, [8], [10]). A counterpart of Ng's result was obtained in [10], where the authors proved the following theorem.

Theorem 24. *Let $I \subset \mathbb{R}$ be an interval, $n \in \mathbb{N}$, and let $f : I \to \mathbb{R}$ be a function.Then f is an n-Wright-convex function if and only if is of the form*

$$f(x) = C(x) + P(x),$$

where C is an n-convex function and P is a polynomial function of degree at most n such that $P(\mathbb{Q}) = \{0\}$. Furthermore, this decomposition is unique.

A polynomial function of order n is the solution of the functional equation

$$\Delta_{a_1,\dots,a_{n+1}} f(x) = 0$$

and the condition $P(\mathbb{Q}) = \{0\}$ was needed to obtain the uniqueness of the decomposition. In view of the previous corollary, we clearly have the following result.

Corollary 25. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and let n be a positive integer. Then if for every $a > 0$ the function $\Delta_a f$ is n-convex or n-concave, then f is $(n+1)$-Wright convex or $(n+1)$-Wright concave.*

However, it is also clear that the converse result is not true because of the presence of higher-order monomials in the polynomial P.

Motivated by the above results, we can define a new class of n-type Wright convex functions. Let n be a positive integer. The n-type Wright convex (concave) functions can be defined as functions such that the function

$$x \mapsto \Delta_a f(x)$$

is convex (concave) of order $(n-1)$ for every $a > 0$.

Proposition 26. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and let n be a positive integer. Then the following statements are equivalent:*

a) the function f is n-type Wright convex (concave),
b) the function f is of the form

$$f = g + A,$$

where A is an additive function and g is a n-convex (n-concave) function.

Corollary 27. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and let n be a positive integer. Then the following statements are equivalent:*

a) for every $a > 0$ the function $\Delta_a f$ is n-convex or n-concave,
b) the function f is $(n+1)$-type Wright convex (concave).

Remark 28. Note that the classes of 1-type Wright convex functions and Wright convex functions coincide, but for $n \geq 2$ we only have that each n-type Wright convex function is a Wright convex function of order n, and the opposite implication is false (cf. Corollary 25).

Summary

As we have shown, functions of the form

$$x \mapsto \Delta_a f(x)$$

for different values of a are strongly connected. If for every a they are n-convex or n-concave then their kinds of convexity are of the same type. Thus, we have extended the understanding of the behavior of the functions generated by the difference operator. It seems that there is still much room for research, since further properties of such functions may also be studied (other than the higher-order convexity).

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